

Ion Diffusion in the Presence of Electric Field

Membrane Potentials

There can exist potential differences between different parts of living cells and the environment. Some of these potentials are shown schematically in [Fig. 1](#).

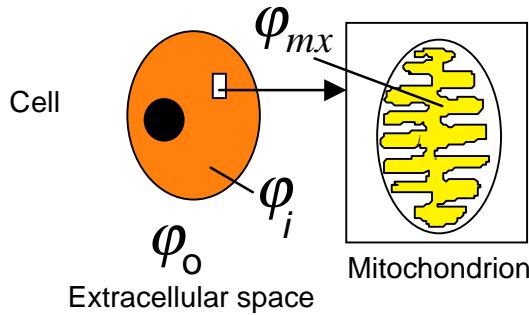


Fig. 1. Some electric potentials inside a living cell.

φ_0 - extracellular potential;
 φ_i - intracellular potential;
 φ_{mx} - potential inside the matrix of mitochondria.

There are potential differences between water phases separated by membranes; these differences are called *transmembrane* or merely *membrane potentials*. A cell membrane potential can be determined as a difference of potentials between intracellular contents φ_i and the environment φ_0 . A mitochondrial potential is a difference of potentials between the matrix of mitochondria and intracellular medium. Thus,

$$\varphi_m = \varphi_i - \varphi_0; \varphi_{mx} = \varphi_x - \varphi_i \tag{1}$$

where φ_m - cell membrane potential, and φ_{mx} - membrane potential of mitochondria.

Besides the transmembrane potential difference, there can exist a difference of electric potentials between membrane's lipid phase and washing aqueous solution, the so-called *interphase potential*. If there are charged chemical groups on membrane's surface (phosphoric acid residues, for instance), a difference of potentials arises between the membrane's surface and the environment, the so-called *surface potential*. Interphase and surface potentials will be considered in detail below, and now we are going to consider in what way the existence on membrane of transmembrane potential will influence ion transfer.

[Fig. 2](#) shows schematically the profile of electric potential in a homogeneous lipid membrane surrounded by salt solutions. Due to high conductivity of saline solutions, the total difference of potentials in the system drops on the membrane's lipid layer which is a very good electric isolator. Inside the membrane (if it is homogeneous by its properties), the electric potential drops linearly, as

it is presented in [Fig. 2](#). This means that the value $\frac{d\varphi}{dx} = -E_x$ is constant inside the membrane.

(This problem will be considered more thoroughly in the next section). The energy of ion, in the process of its motion across the membrane, changes by (according to) the same linear law.

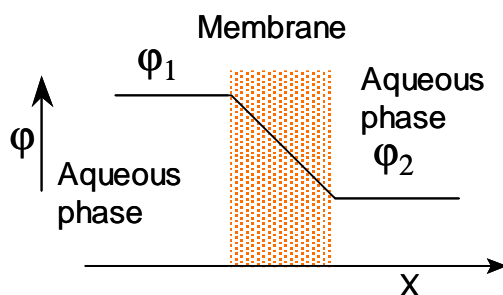


Fig. 2. Profile of potential in homogeneous membrane.

This reasoning, however, is based on the approach that membrane is a completely homogeneous body; but the lipid part of the membrane consists of two layers of phospholipid molecules, and the sizes of mobile links of fatty acid chains in these molecules are commensurable with the sizes of the ions travelling inside the membrane. This fact makes one reject a completely macroscopic approach to the phenomena while considering ion transfer in membrane and consider the process on the microscopic (i.e. molecular) level.

Changes of potential barrier sizes for ion motion in the presence of external electric field

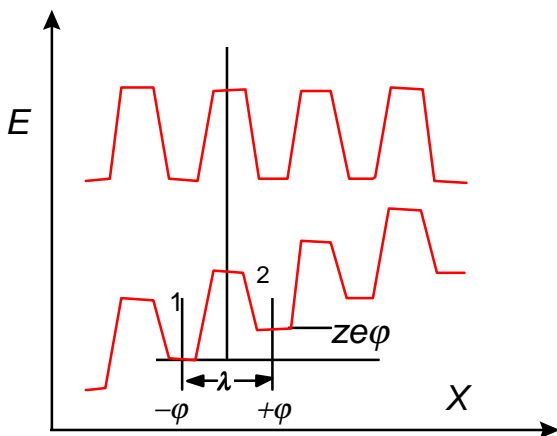
In the absence of electric field in a homogeneous membrane, the height of barriers and the depth of pits are everywhere the same. This is why the probability of accidental jumping of ion from left to right is equal to the probability of its jumping from right to left (see Sections 3 and 4); but this pattern alters if electric field is applied to the membrane (see Fig. 3).

It is known from electrostatics that the existence of the field E_x directed along the X-axis means that the potential along this axis changes:

$$E_x = -\frac{\partial\phi}{\partial x} \tag{2}$$

This, in its turn, means that the ion's energy changes depending on the coordinate of the pit or barrier where the ion is (present) at a given moment. If the potential at some site is equal, ϕ , then this adds to the ion energy a summand equal to $ze\phi$, where ze is ion's electric charge, and e - elementary charge, i.e. proton's charge. The value z is the ion's *dimensionless (nondimensional) charge*.

Fig. 3. Changes in the energetic profile of the membrane in the presence of electric field.



In the presence of the *field* the energy of cations grows with the growth of potential (scheme below).

Vice versa, the energy of anions decreases with the growth of potential (not shown in the figure).

The additional energy conveyed to the ion in the presence of external electric field alters the membrane's energetic profile, as it is shown in Figs. 3 and 4. For positive ions (cations), the level of all barriers and pits grows in the direction of the potential's growth.

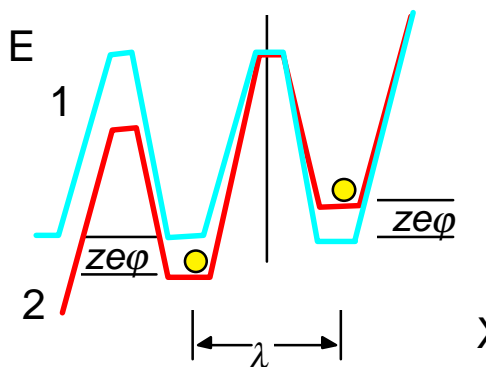


Fig. 4. Changes in the ion energy profile in membrane in the absence (1) and in the presence (2) of electric field.

Potential differences between every pit and neighbouring barriers are $ze\phi$. For details see the text.

Designate as $\Delta\phi$ the difference of potentials between a pit and a neighbouring left barrier. If the pits and barriers across all the membrane are similar and symmetrical the difference of potentials between each barrier and the neighbouring left pit will also be equal.

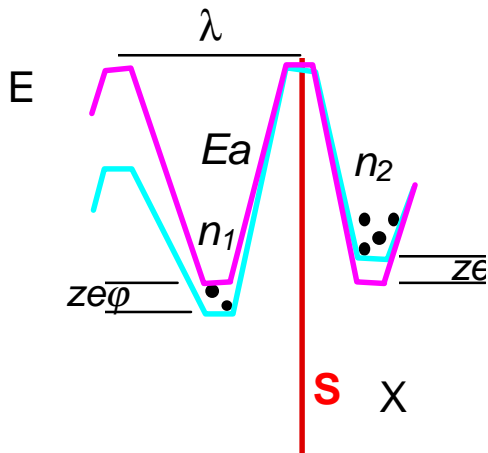


Fig. 5. Ion flux (flows) through barrier.

n_1 - ion concentration to the left of plane S in volume S; n_2 - the same to the right of plane S. The ion flow from left to right is proportional to n_1 and depends on the height of the barrier to the right (activation energy is $E + ze$). The ion flow from right to left is proportional to n_2 and depends on the height of the barrier to the left of the pit (activation energy is $E - ze$).

Assume that the ion energy in the potential pit was initially equal to E , and the potential was equal to 0. In the presence of a potential in a neighbouring pit, the ion's energy in it will be higher by ze , where e is elementary charge, i.e. proton's charge, and z - dimensionless ion's charge (for K^+ or Na^+ , $z = 1$; for Cl^- $z = -1$; for Ca^{2+} $z = 2$, etc.). It means that if an electric field is applied to a membrane, such that the created potential difference between two pits is equal to $2ze$, then the frequency of ion jumps from the pit to the left increases (for positive ions) and becomes equal to (3),

$$v = v_0 e^{\frac{E - ze\phi}{kT}}, \quad (3)$$

and the frequency of jumps of (positive) ion to the right decreases and becomes equal to:

$$v = v_0 e^{-\frac{E + ze\phi}{kT}} \quad (4)$$

(See Fig. 5). If even the quantity of ions in all the pits is the same (i.e. the summary diffusional flow of the particles is equal to 0), the prevailing motion of cations (flow of ions) from right to left will take place in the membrane due to the different probability of jumping of each ion to the left and right.

The motion of ions in magnetic field is termed as *electrophoresis* (see Fig. 6, centre).

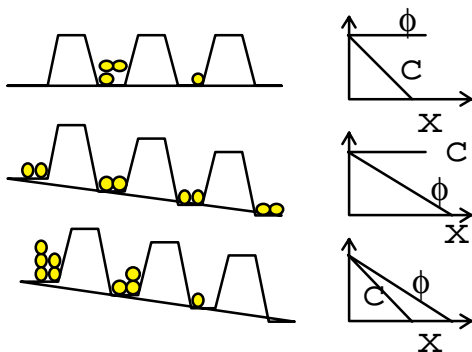


Fig. 6. Membrane's energetic profiles (to the left), and changes in potential through the membrane's thickness and in ion concentration C .

Above - the absence of the gradients of concentration and potential on the membrane; below - the presence of the gradients of concentration and potential.

In a more case, the motion of ions by the laws diffusion (to the direction of a lower concentration) and due to electrophoresis are combined; thus, one can speak about the *electric diffusion of ions* (see [Fig. 6, below](#)).

To consider flows during electric diffusion, it is necessary to draw plane S through the membrane in such a way that it would pass across the middle between two neighbouring energetic pits. A flow of ions from left to right will be equal to the quantity of ions (N_1) in a volume S to the left of plane S multiplied by the frequency of jumps to the right ($v_0/2$) and multiplied by the probability of a "successful" jump: (5)

$$\Phi_{n \rightarrow} = N_1 \frac{v_0}{2} e^{-\frac{E+e\phi}{kT}} \quad (5)$$

In a similar way, the flow of ions to the left will be the following:

$$\Phi_{n \leftarrow} = N_2 \frac{v_0}{2} e^{-\frac{E-ze\phi}{kT}}, \quad (6)$$

where N_2 - the number of ions in a volume S to the right of plane S.

The number of ions (N) in each case is equal to the product of ion concentration (n particles/m³) and into the volume S i.e. $N = nS$.

The summary flow to the X-axis direction can be expressed by the following formula:

$$\Phi_n = S\lambda \frac{v_0}{2} \left(n_1 e^{-\frac{E+ze\phi}{kT}} - n_2 e^{-\frac{E-ze\phi}{kT}} \right), \quad (7)$$

and the flow density

$$J_n = \lambda \frac{v_0}{2} \left(n_1 e^{-\frac{E-ze\phi}{kT}} - n_2 e^{-\frac{E+ze\phi}{kT}} \right). \quad (8)$$

To simplify this or many following equations the *dimensionless potential* value should be introduced:

$$\psi = \frac{e}{kT} \phi \quad (9)$$

Besides, The value $e^{-\frac{E}{kT}}$ is taken out of the brackets. To simplify records, the dimensionless energy value should be used:

$$W = \frac{E}{kT} \quad (10)$$

Now Eq. 8 can be recorded as follows (acquires the following form):

$$J = \lambda \frac{v_0}{2} e^{-W} \left(n_1 e^{-z\psi} - n_2 e^{+z\psi} \right). \quad (11)$$

One-barrier model of ion transport

Local ion concentrations in membrane's microareas (n_1 and n_2) as well as local potential changes can not be determined experimentally, this is why Eq. 11 can be of little use. The purpose of its further transformations is to obtain, in the right part of the equation, real experimental values of transmembrane potential and ion concentrations in water phase. Such transformations in theoretical biophysics are carried out with the use of these or those assumptions of the internal structure of membranes. Below, two extreme cases are considered:

- 1) There is one barrier for ions in the center of the membrane (the unibarrier model of ion transport).
- 2) The number of barriers in the membrane is very large, and they all are equal in sizer in the absence of field (the multibarrier model).

One-barrier model

The ions in the well to the membrane's left are in the lipid phase and they are distributed between this phase and the neighboring water phase. So, ion concentrations in the «wells» ($C_{m1} = n_1$ and $C_{m2} = n_2$) can be expressed through their concentrations in the water phase (C_1 and C_2) using the distribution coefficient:(13).

$$K = \frac{C_{m1}}{C_1} = \frac{C_{m2}}{C_2}, \quad (12)$$

$$J = \lambda \frac{v_o}{2} e^{-W} K (C_1 e^{-z\psi} - C_2 e^{+z\psi}). \quad (13)$$

Substituting the expression $\frac{\lambda^2 v_o}{2} e^{-W}$ for the diffusion coefficient D (compare Eq. 6 in Section 4), we obtain the following expression:

$$J = \frac{DK}{\lambda} (C_1 e^{-z\psi} - C_2 e^{+z\psi}). \quad (14)$$

In our case, the value λ (ion displacement along the X-axis at every jump) is equal to the membrane's thickness (l). This allows to substitute DK/λ in Eq. 14 for the permeability coefficient $P = DK/\lambda$. The final equation of the flow under the unibarrier model of ion transfer looks like that:

$$J = P(C_1 e^{-z\psi} - C_2 e^{+z\psi}) \quad (15)$$

Under the unibarrier model of membrane transport, the transmembrane potential ψ_m is equal to the potential difference between two neighbouring «wells»; thus, $\psi = (\psi_m/2)$ in Eq. 15 where ψ_m is transmembrane potential in the dimensionless form.

Electric diffusion of ion in homogeneous medium

Below, the *multi-barrier* model of ion transport will be considered. Come back to Eq. 11.

If one proceeds from ion transfer inside the membrane by the kinks mechanism, the simplest assumption is that the membrane is homogeneous and the ion when moving overcomes *many* identical wells separated by barriers of the same height. The potential difference between neighboring wells is not large, and this allows to substitute e^{-z} and e^{+z} by the approximate values $1 - z$ and $1 + z$:

$$e^{-z} \approx 1 - z; e^{+z} \approx 1 + z \quad (16)$$

Thus, Eq. 11 can be recorded as follows: (17).

$$J = \lambda \frac{v_o}{2} e^{-W} (C_{m1} - C_{m1} z\psi - C_{m2} - C_{m2} z\psi). \quad (17)$$

After some rearrangement, the following equation is obtained:

$$J = \lambda \frac{v_o}{2} e^{-W} [C_{m1} - C_{m2} - z\psi(C_{m1} + C_{m2})]. \quad (18)$$

At small values, it can be considered that (19).

$$\frac{C_{m2} - C_{m1}}{\lambda} = \frac{dC_m}{dx} \quad (19)$$

In the same way, the potential difference between neighbouring barriers (as well as between neighbouring wells) the distance between which is λ , is related with the gradient of potential along the X-axis by an obvious relation (ratio):

$$\frac{2\psi}{\lambda} = \frac{d\psi}{dx} \quad (20)$$

Moreover, it is useful to introduce the concept of mean ion concentration in the region of a given well: (21).

$$C_m = \frac{C_{m1} + C_{m2}}{2} \quad (21)$$

Substituting these values into Eq. 18, the following equation is obtained: (22).

$$J = -\lambda^2 \frac{v_o}{2} e^{-W} \left(\frac{dC_m}{dx} + zC_m \frac{d\psi}{dx} \right). \quad (22)$$

After that we substitute into Eq. 22 the diffusion coefficient (see Eq. 6 in Section 4):

$$J = -D \left(\frac{dC_m}{dx} + zC_m \frac{d\psi}{dx} \right). \quad (23)$$

Thus, we have deduced a very important equation which is called «*the basic equation of electrodiffusion*».

It should be stressed that inspite of the fact that the electrodiffusion equation was derived for a concrete case - the case of ion motion in the lipid phase of membranes, it can be applied to any homogeneous medium including water solutions. That is why it is often written without the concentration index (i.e. C instead of C_m).

Diffusion and electrophoresis

It is very interesting to analyze Eq. 23 for two particular cases.

Case 1. Particles are not charged (neutral molecules), or there is no electric field across the membrane (transmembrane potential difference is absent). In this case $z = 0$, or $\frac{d\psi}{dx} = 0$, or both these values are equal to 0. In all the variants, the second summand (component) in brackets are equal to 0; thus, we obtain the already known Fick's law for molecule diffusion in homogeneous medium:

$$J = -D \frac{dC}{dx}. \quad (24)$$

Case 2. There is no ion concentration gradient inside the membrane (it usually means that ion concentrations in water phase are equal on both membrane sides. In this case the first summand in brackets in Eq. 26 is equal to 0, and the ion flow can be expressed by the following formula:

$$J = -DzC \frac{d\phi}{dx} = -DzC \frac{e}{kT} \frac{d\phi}{dx} = C \frac{D}{kT} zeE_x, \quad (25)$$

where E_x - electric field intensity in the X direction. The product of the field and charge is equal to a force F acting on the particle, i.e. $F = zeE_x$. Thus, there exists a direct proportionality between the electric force acting on every particle and the density of the flow of particles moving under the action of this force:

$$J = CuF, \quad (26)$$

where the proportionality coefficient $u=(D/kT)$ is called the ion *electrophoretic mobility*. It is not difficult to see that the ion mobility is proportional to the diffusion coefficient.

Electrodiffusion as the sum of diffusion and electrophoresis

The introduction of electrophoretic mobility allows to make easier the expressions in the electrophoresis equation (25):

$$J = -zeCu \frac{d\phi}{dx}, \quad (27)$$

and to change the form of writing of the basic electrodiffusion equation (23)

$$J = -D \frac{dC}{dx} - zeCu \frac{d\phi}{dx}. \quad (28)$$

This equation is known as the *Nernst-Planck equation*. The comparison of this equation with Fick's equation for diffusion (24) and electrophoresis equation (27) shows that the summary flow in the case of electrodiffusion is added algebraically of diffusion and electrophoretic flows; in other words, diffusional and electrophoretic ion motions occur irrespective of each other.

Ion motion rate in electric field

The relationship between the flow density of ions moving under the action of electric field, and the motion rate of each particle is very simple. See [Fig. 7](#).

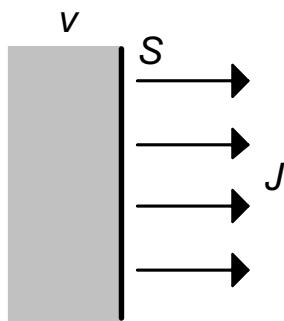


Fig. 7. Relationship between flow density (J) and the rate of motion of each particle (v).

Since vSC kilomoles of particles (C - molar concentration) pass through square S every second, the flow = vSC , and the flow density $J = vC$.

Assume that ions move through a plane S in the X direction under the action of electric field. Every second each ion passes a distance $1v$, m, where v - ion transfer rate, m/sec. Laying off this distance to the left of plane S , we obtain a volume $1vS$ which contains $1vSn$ particles, or $1vSC$ kilomoles of substance. This amount of substance is carried for one second through plane S . Thus, the flow

$$\Phi = vSC, \quad (29)$$

and the flow density is equal to the product of the transfer rate of particles and their molar concentration:

$$J = vC, \quad (30)$$

Relationship between ion flow and electric current in medium

Since each ion carries a charge equal to ze , a simple relationship exists between the particle flow density (J) ($c^{-1} \cdot m^{-2}$) and the density of electric current (j), $A \cdot m^{-2}$:

$$j = zeJ, \quad (31)$$

If ion concentration in the medium is uniform (i.e. the concentration gradient is equal to 0), the equation for current density may be derived from Eqs. 27 and 31:

$$j = -z^2 e^2 C u \frac{d\phi}{dx}, \quad (32)$$

This equation can be compared to the well-known Ohm's law for continuous media:

$$j = -\sigma \frac{d\phi}{dx}, \quad (33)$$

where σ , Ohm^{-1} - medium conductivity. Thus, the used above theory of random walks with regard for changes in the energetic profile in electric field not only allows to deduce Ohm's law but exposes the matter of conductivity if it is conditioned by the transfer of one ion:

$$\sigma = -z^2 e^2 C u, \quad (34)$$

If a medium (whether it is membrane lipid layer or water solutions) contains several ions, the total electroconductivity is the sum of conductivities of all ions:

$$\sigma = e^2 \sum_i z_i^2 C_i u_i, \quad (35)$$

Questions

- 1) Membrane potentials.
- 2) Alterations of potential barriers for ion motion in the presence of external electric field.
- 3) Unibarrier model of ion transport.
- 4) Electric diffusion of ion in homogeneous medium.
- 5) Diffusion and electrophoresis.
- 6) Relationship between ion flow and electric current in medium.